REVIEW

from Associate Professor Sofiya Ivanovska,
Institute of Communication and Information Technologies
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of the Ph.D. thesis of

Silvi-Maria Todorova Gurova

with the title

"Stochastic Numerical Methods for Eigenvalue Estimation"

submitted for the acquisition of the educational and scientific degree of "Doctor" in the doctoral program "Mathematical modeling and application of mathematics" professional field 4.5 "Mathematics" with advisor Prof. Aneta Karaivanova, Ph.D., IICT-BAS

According to a decision of the Scientific Council of IICT (protocol № 4/29.04.2026) and order № 101/04.05.2026 of the director of IICT, I have been appointed as a member of the scientific jury for the procedure for acquiring the educational and scientific degree "Doctor" in the doctoral program "Mathematical modeling and application of mathematics" in professional field 4.5 Mathematics. In accordance with the decision of the first meeting of the jury (protocol № 1/8.05.2026), I have been selected as a reviewer. I received the documents for the procedure in accordance with the relevant regulatory framework.

1. Relevance of the topic

The estimation of extreme eigenvalues of symmetric matrices is a problem of growing relevance, because it finds application in a number of modern scientific and technological fields - from quantum mechanics and financial modeling to machine learning and data analysis. With the continuous increase in the dimensionality of problems in these fields, traditional exact and iterative methods prove to be not efficient enough due to limitations in terms of available memory, computational complexity and difficulty in parallelization. Stochastic methods are emerging as a promising alternative, as they are naturally parallelizable and scalable to matrices with thousands or millions of elements. The development of high-performance computing systems further increases the need for new, faster and more accurate stochastic algorithms, adapted to modern hardware. This is exactly what makes research in this area not only theoretically significant, but also practically urgent in the context of the current challenges of science and industry.

2. Knowledge of the state of the topic

The candidate demonstrates a thorough knowledge of the topic of the present multidisciplinary research and good awareness of the current state of the problem under consideration. This is confirmed by the clearly formulated objective, the correctly defined research tasks, the coherent presentation in the dissertation and the bibliography used, which includes 121 cited sources.

3. Research methodology

The research is focused on the estimation of extreme eigenvalues of non-singular symmetric matrices, formulated through the standard equation $Ax=\lambda(A)x$. For finding the maximum eigenvalue, the Power and Quasi-Monte Carlo methods are applied (based on the Power and Resolvent Power method) and for the minimum - the Resolvent Monte Carlo and Resolvent Quasi-Monte Carlo approach. In Monte Carlo methods, a random variable is constructed with an expected value equal to the sought solution, where the approximate value is obtained as the arithmetic mean of N independent realizations. In this case, the probabilistic error is of the order of $O(N^{-1/2})$, according to the Central Limit Theorem. In Quasi-Monte Carlo methods, instead of pseudo-random points, deterministic sequences with low discrepancy are used (Sobol and Halton sequences), where the error, estimated through the Koksma-Hlawka inequality, is of a better order $O(N^{-1}(\log N)^k)$.

In order to overcome the main drawback of deterministic quasi-random sequences, namely the lack of a natural probabilistic approach for error estimation and the risk of correlations at higher dimensionalities, randomized (scrambled) Sobol and Halton sequences are used in the dissertation. The scrambling is implemented through a combination of the *MatousekAffineOwen* and *RR2* methods, together with the *skip* and *leap* operators, which reduces correlations and improves uniformity of coverage. The iterative stochastic methods, built on a Markov chain with k steps (transitions), combine systematic error (depending on the number of iterations) and stochastic error (depending on the sample size N), where a key goal of the research is finding the optimal balance between the two types of errors.

4. Structure and main contributions of the dissertation

The dissertation of Silvi-Maria Gurova has a total volume of 119 pages. Included are introduction, three main chapters, conclusion, appendix and bibliography of 121 sources. It contains 15 figures, 16 tables and a list of notations and abbreviations used in the text. The main goal of the dissertation is to propose and analyze stochastic numerical methods for estimating eigenvalues through Monte Carlo and randomized quasi-Monte Carlo algorithms for approximate computation of the extreme eigenvalues of symmetric square matrices. The conclusion lists the main scientific and applied scientific contributions, which are summarized below.

The introduction begins with presentation of the significance of the problem of finding extreme eigenvalues of large-dimensional square matrices and its applications in quantum mechanics, financial mathematics, spectral graph theory, image processing and tensor analysis. A detailed literature review is presented, covering the main scientific achievements of leading researchers in the field under consideration, which demonstrates the PhD student's good awareness of the current state of the problem. It is defined according to the equation $Ax=\lambda(A)x$, where the minimum and maximum eigenvalue are sought according to the ordering $\lambda_{min} \leq \ldots \leq \lambda_{max}$. Iterative methods with computational complexity $O(n^2k)$ are examined, together with stochastic methods that overcome the limitations of deterministic iterative methods.

Stochastic methods are characterized by a linear relationship between the dimensionality of the problem and the required memory, easy parallelization and applicability to large-dimensional matrices. Standard Monte Carlo achieves an error of order $O(N^{-1}/^2)$, however quasi-Monte Carlo methods achieve $O(N^{-1}(\log N)^k)$, which represents a significant improvement for smooth integrands. The Sobol and Halton sequences and their randomization through the *skip* and *leap* operators are described, which reduce correlations at high dimensionalities. Four tasks are formulated: investigation of the Power (Resolvent) Monte Carlo and quasi-Monte Carlo methods, development of efficient algorithms, conducting numerical experiments, and application to a problem from financial mathematics for market risk assessment. The experiments were carried out on 12 servers Fujitsu Primergy RX 2540 M4 with GPU cards NVIDIA Tesla V100 32GB and processors Intel Xeon Gold 5118.

The first chapter describes the development and investigation of efficient stochastic algorithms for estimating the maximum eigenvalue of symmetric matrices using the Power Monte Carlo and Power Quasi-Monte Carlo methods. At the beginning of the chapter the problem of finding the maximum eigenvalue λ_{max} of a non-singular symmetric matrix is formulated through the equation $Ax=\lambda(A)x$. The deterministic Power method is presented, based on the Rayleigh quotient and its convergence is explained, determined by the ratio $|\lambda_2/\lambda_1|^2$, where it is noted that difficulties arise when the first two eigenvalues are close. Then the Stochastic Power method is introduced, in which the maximum eigenvalue is expressed as a limit of a ratio of scalar products, estimated through a Markov chain. The random variables $\theta^{(k)}$ and $\theta^{(k-1)}$ are constructed, whose expected values are equal to the corresponding scalar products and their ratio gives an approximation of λ_{max}. The main role is played by the choice of an admissible initial density vector and an admissible transition density matrix (equation (1.16)). On this basis, the nearly optimal PMC and PQMC algorithms are developed, presented with pseudocode, with computational complexity $O(Nnk+n^2)$, where for $N>n$ the term $O(Nnk)$ dominates. For the PQMC variant, scrambled Sobol and Halton sequences are used instead of pseudo-random number generators. The numerical experiments are conducted on two symmetric dense matrices with dimensions $n=100$ and $n=500$, using the Mersenne Twister and Middle Square generators in the case of PMC algorithms. The results show that the nearly optimal algorithms outperform their classical counterparts in terms of accuracy and computational efficiency, with the variance in the nearly optimal variants being about two orders of magnitude smaller compared to the classical ones. The absolute errors in the nearly optimal PMC algorithms are of the order of 10^{-4} to 10^{-5}, while in the case of the classical variants they are 10^{-2}. Optimal balance between the stochastic and systematic error is achieved at k=8 for n=100 and k=9 for n=500 under PMC, and respectively k=11 and k=12 for PQMC with Sobol sequences, with the algorithms using Sobol sequences outperforming these with Halton sequences.

In the second chapter, based on the Resolvent Monte Carlo method proposed by Karaivanova and Dimov, a nearly optimal Monte Carlo approach is described for estimating the minimum eigenvalue of a symmetric matrix, which avoids the explicit computation of the inverse matrix, that is a costly operation for large-dimensional problems. Instead of the inverse matrix, a resolvent matrix R_q is introduced which is represented as an infinite power series and whose eigenvalues are related to those of A through an explicit formula. The sign of the acceleration parameter q determines which extreme eigenvalue is being estimated, where for the present research the case of primary interest is $q<0$. The convergence of the Resolvent Power method depends significantly on the two parameters q and m, where a larger m accelerates convergence, however at higher computational cost. In practice, the resolvent matrix is approximated by a finite sum, truncated at step k, where the systematic error from this truncation is controlled analytically. A key theoretical result is the established linear relationship between the parameters m and k, which allows for an optimal choice of the pair (m, k) for a given level of systematic error ε and acceleration parameter q. In a stochastic context, the expected values are approximated through sample means from N realizations of a Markov chain, where the two proposed algorithms, Resolvent Monte Carlo and Resolvent Quasi-Monte Carlo differ in the type of sequences used. In the Monte Carlo approach, the stochastic error is of the order of $O(N^{-1/2})$, while in the quasi-Monte Carlo approach faster convergence can be expected through the use of low-discrepancy sequences. To add controlled randomness without losing the advantages of deterministic sequences, randomized Sobol sequences are used. The numerical experiments are conducted on randomly generated symmetric matrices with dimensions n=500 and n=1000, where the eigenvalues are constructed to guarantee convergence of the method. The parameters m and k are not chosen arbitrarily, but are determined based on the theoretical estimates for the systematic error at a pre-fixed level $\varepsilon_{sys}=0.001$. If n=500 the best value of the balance between the two types of errors is achieved when k=5, m=10 and N=512, and if n=1000 - at k=5, m=5 and N=1024, which confirms the theoretical assumption about the linear relationship between m and k. It is established that increasing N beyond this threshold does not lead to monotonic improvement in accuracy, since the systematic error, controlled by m and k, already determines a lower bound for the achievable accuracy. This highlights that the total error depends on the balance between its two types and not only on the number of simulations.

The third chapter applies the developed stochastic algorithms to a real financial problem - estimating the concentration of market risk in an investment portfolio through spectral analysis of correlation matrices. The correlation matrix is constructed from empirical financial data on asset returns, where its maximum eigenvalue serves as a quantitative indicator of the degree to which the joint movement of assets is explained by one dominant market factor. This value is directly related to the fraction of variance explained (FVE_1), which measures what portion of the total variability of the portfolio is absorbed by the first principal component. The data are extracted from two independent sources with different temporal and structural characteristics, allowing for a comparison of algorithmic behavior at different dimensionality and information density. The first matrix covers long-term monthly observations on 32 global assets, and the second - five-year data on 490 companies from the S&P500 index. For the first matrix the fraction of variance explained is FVE_1=56.71% and for the second - FVE_1=41.04%, which shows that for the broader market index the risk is more distributed among multiple factors. The numerical experiments reveal that for both matrices there exists an optimal value of the number of transitions k, at which a balance between the systematic and stochastic error is achieved, and that further increasing k does not significantly improve accuracy. This balancing value is preserved for both matrices regardless of the difference in their dimensionality, which confirms the robustness of the theoretical estimates. For the larger portfolio, with a small number of simulations the algorithms with *skip* and *leap* parameters show an advantage, but it weakens with larger N. For larger N, the PMC(MT) and PQMC(d)(S) algorithms show an advantage, demonstrating stable and monotonic convergence. The final conclusion is that to achieve operational accuracy of the order of a few decimal digits, a relatively small number of simulations is sufficient, which makes the algorithms practically applicable for market risk estimation with large-dimensional matrices.

The scientific contributions of the dissertation can be summarized as follows:

- The construction of the random variables $\theta^{(k)}$ and $\theta^{(k-1)}$ has been refined for the case of using a nearly optimal transition density for the Markov chain and it has been proven that for certain classes of symmetric matrices this construction simplifies significantly.
- A linear relationship between the parameters m and k in the Resolvent method has been established, which allows for their optimal selection at a pre-fixed level of systematic error.

The applied scientific contributions of the dissertation can be summarized as follows:

- Nearly optimal Power and Resolvent Monte Carlo and quasi-Monte Carlo algorithms have been developed for estimating extreme eigenvalues of symmetric matrices, described with pseudocode and with given conditions for error balancing.
- It has been established that the use of randomized Sobol sequences provides systematically better accuracy and convergence compared to Halton sequences and pseudo-random number generators at the same computational complexity.
- An implementation of the Resolvent quasi-Monte Carlo algorithm has been created, which allows efficient estimation of the minimum eigenvalue of large-dimensional matrices.

- The developed algorithms have been applied to a real problem from financial mathematics for estimating the market risk of an investment portfolio through the maximum eigenvalue of the correlation matrix of the assets.
- It has been shown that the fraction of variance explained FVE_1 can be computed with high accuracy using a relatively small number of simulations, which makes the method applicable for practical financial analysis under conditions of growing dimensionality.

Silvi-Maria Gurova has presented four publications, related to her dissertation. One of them is published in the scientific journal Contemporary Mathematics. Two are articles in the proceedings of the Large-Scale Scientific Computing conference. Two of the publications are indexed in Scopus and Web of Science, and the third is accepted for publication at Lecture Notes in Computer Science and will also be indexed. Furthermore, the candidate has presented a short communication at the 4th International Conference Numerical Methods for Scientific Computations and Advanced Applications in co-authorship with her thesis advisor. Silvi-Maria Gurova is the first author in all publications. The results shown in the dissertation have been presented at 5 scientific conferences.

5. Assessment of compliance with the minimum requirements according to the regulatory framework

The candidate satisfies the minimum national requirements in the professional field and no plagiarism has been detected in the scientific works submitted as part of the procedure, nor in the dissertation, as she has provided a report from a check of the dissertation using a licensed system for detection of matching texts.

6. Author summary

The author summary is 51 pages long and includes 94 citations. It reflects the content of the dissertation and gives a clear picture of the achieved results and contributions of the dissertation.

7. Notes and recommendations

My notes and recommendations on the preliminary text have been fully taken into account in the final version of the dissertation. I noticed a typo on page 102, where the mentioned IF: 2.5 (2025) should be IF: 2.5 (2024). As a continuation of this work, I would recommend investigating the possibilities for a systematic selection of the *skip* and *leap* parameters. The development of an adaptive criterion for their selection based on the characteristics of the problem would improve the efficiency and practical applicability of the algorithms.

8. Personal impressions of the candidate

I have known Silvi-Maria Gurova professionally since she joined the institute. We have worked together on various projects and she approaches the assigned tasks diligently. The candidate shows initiative and a desire to develop in her scientific work. She strives to acquire new knowledge and shows interest in research activities and commitment to fulfilling her academic duties. She demonstrates a willingness to take into account suggestions and guidance from colleagues and to apply them in her work.

9. Conclusion

Based on everything stated above, I consider that the presented materials satisfy all the requirements of the Law on the Development of the Academic Staff of the Republic of Bulgaria (LDASRB), the Regulations for the Implementation of the LDASRB, the BAS Rules for the Implementation of the LDASRB as well as the Regulations for the Specific Conditions for Acquiring Scientific Degrees and for Holding Academic Positions in IICT-BAS.

I give my positive assessment for awarding Silvi-Maria Todorova Gurova the educational and scientific degree of "Doctor" in the doctoral program "Mathematical modeling and application of mathematics" in professional field 4.5 "Mathematics".

НА ОСНОВАНИЕ
ЗЗЛД

29.05.2026
Sofia